Two-dimensional quantum spaces corresponding to solutions of the Yang-Baxter equations



1991 J. Phys. A: Math. Gen. 24 2903

(http://iopscience.iop.org/0305-4470/24/13/010)

J. Phys. A: Math. Gen. 24 (1991) 2903–2912. Printed in the UK

# Two-dimensional quantum spaces corresponding to solutions of the Yang–Baxter equations

Ladislav Hlavatý

Institute of Physics, Czechoslovak Academy of Sciences, Na Slovance 2, 180 40 Prague 8, Czechoslovakia

Received 25 February 1991

**Abstract.** The quantum spaces given by relation $f(\hat{R})(x \otimes x) = 0$, where $R$ are eight- (and less) vertex constant solutions of the Yang–Baxter equations, and their invariance algebras are investigated.

## 1. Introduction

The supersymmetric spacetimes are Grassman algebras with odd and even elements. Recently new algebras called *quantum vector spaces* that are deformations of the Grassman algebras, were introduced into mathematical physics [1]. They are defined as factor algebras of the associative algebra generated by $n$ generators $x_1, x_2, \ldots, x_n$ divided by an ideal corresponding to some relations in $\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle$.

A method for constructing Hopf algebras that represents a deformation of classical Lie algebras and some other related structures was also suggested [2]. The principal parameter needed for the construction is a matrix $R = \{R_{ij}^{kl}\}$ that satisfies the constant Yang–Baxter equations (YBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.1}$$

In this paper we are going to deal with quantum spaces given by the following *quadratic relationships:*

$$f(\hat{R})_{ij}^{kl}x_k x_l = 0 \qquad \text{symbolically } f(\hat{R})(x \otimes x) = 0 \tag{1.2}$$

where $f$ is a polynomial function of

$$\hat{R} := PR \qquad P(x \otimes y) := (y \otimes x).$$

If $R$ satisfies the YBE then $\hat{R}$ is a representation of the braid group

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{1.3}$$

It is clear that if $\det f(\hat{R}) \neq 0$ then (1.2) implies $x_i x_j = 0$ for all $i, j$. However, for the zero polynomial there is no relationship. We shall call these two cases *trivial* quantum spaces and we shall consider only singular non-zero polynomials $f$.

0305-4470/91/132903+10$03.50  

Let us define the (matrix) algebra $A_R$ as a free algebra generated by $n^2$ elements $T_i^j$, $i,j = 1,\ldots,n$, factorized by the following relationships:

$$\hat{R}_{ij}^{kl} T_k^m T_l^n = T_i^k T_j^l \hat{R}_{kl}^{mn} \qquad \text{or} \qquad \hat{R}(T\otimes T) = (T\otimes T)\hat{R}. \tag{1.4}$$

The transformations $x_i' := T_i^j x_j$ where the $T_i^j$ satisfy (1.4) then represent homomorphisms of quantum spaces given by (1.2). In other words, the quantum spaces are left comodules of $A_R$ [1, 2]. For these reasons we shall call $A_R$ *the invariance algebra* of all the quantum spaces corresponding to $\hat{R}$.

The complete list of eight- (and less) vertex constant solutions of the YBE was given in [3]. Later we are going to present non-trivial quantum spaces with $n = 2$ that follow from these solutions and their invariance algebras. The numbering of the solutions is identical to that in [3] and $q, r, s$ and $t$ are arbitrary complex constants different from zero. Instead of solutions of the YBE we display here the corresponding braid group representations $\hat{R} = PR$.

As the $\hat{R}$-matrices we consider are $4\times 4$, they satisfy polynomial equations of maximally fourth order. Therefore we can consider maximally third-degree polynomials in (1.2). Moreover, the minimal polynomials for most of them is a degree less than four. This essentially restricts the number of resulting quantum spaces.

As previously mentioned, all quantum spaces corresponding to a fixed $\hat{R}$ are invariant with respect to

$$\delta : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.5}$$

if $A, B, C$ and $D$ satisfy the quadratic relationships given in (1.4). However, in many cases these relationships can be obtained as necessary conditions for invariance of a set of quantum spaces under (1.5) (cf [1]).

## 2. List of the quantum spaces

Let us now investigate the explicit forms of the quantum spaces and their invariance algebras.

*Case 0.* Let $R_0 = P$. Then $\hat{R} = 1$ and it is self-evident that there are no non-zero polynomials with $\det f(\hat{R}) = 0$. Thus only trivial quantum spaces exist for this solution. The algebra $A_R$ is also trivial in the sense that there are no relationships for $A, B, C$ and $D$.

*Case 1.* The matrix $\hat{R}$ that corresponds to the solution $R_1$ from [3] is

$$\hat{R}_1 = \begin{pmatrix} 1 & 0 & 0 & \mathrm{i} \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ \mathrm{i} & 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

The minimal polynomial condition for $\hat{R}_1$ is (the Hecke condition)

$$(\hat{R}_1 - 1 - \mathrm{i})(\hat{R}_1 - 1 + \mathrm{i}) = 0 \tag{2.2}$$

and the only singular matrix polynomials (up to regular factors) are projectors

$$f_+ = -\mathrm{i}(\hat{R}_1 - 1 + \mathrm{i})/2 \qquad f_- = \mathrm{i}(\hat{R}_1 - 1 - \mathrm{i})/2. \tag{2.3}$$

The rank of both of these is two and they define the quadratic relationships

$$x_1^2 + x_2^2 = 0 \qquad x_1x_2 + \mathrm{i}x_2x_1 = 0 \tag{2.4}$$

and

$$x_1^2 - x_2^2 = 0 \qquad x_1x_2 - \mathrm{i}x_2x_1 = 0 \tag{2.5}$$

for the quantum spaces $Q_+(\hat{R}_1)$ and $Q_-(\hat{R}_1)$, respectively. These spaces, or more explicitly, the relationships (2.4) and (2.5) are invariant with respect to the map (1.5) if and only if $A, B, C$ and $D$ satisfy

$$\begin{aligned} &AB = -\mathrm{i}DC \qquad BA = \mathrm{i}CD \qquad AC = \mathrm{i}DB \qquad CA = -\mathrm{i}BD \\ &B^2 = C^2 \qquad A^2 = D^2 \qquad AD = DA \qquad BC = -CB. \end{aligned} \tag{2.6}$$

These relationships are identical to those for the invariance algebra $A_R$ obtained from (1.4) for $\hat{R}_1$.

*Case 2.*

$$\hat{R}_2 = \begin{pmatrix} 1+t & 0 & 0 & 1 \\ 0 & 1 & \sqrt{1+t^2} & 0 \\ 0 & \sqrt{1+t^2} & 1 & 0 \\ 1 & 0 & 0 & 1-t \end{pmatrix} \tag{2.7}$$

(see also [4]). The minimal polynomial condition is

$$(\hat{R}_2 - 1 - \sqrt{1+t^2})(\hat{R}_2 - 1 + \sqrt{1+t^2}) = 0. \tag{2.8}$$

The invariance algebra $A_R$ is given by the relationships following from (1.4)

$$\begin{aligned} &CD = \sqrt{1+t^2}BA - tAB \qquad DC = \sqrt{1+t^2}AB - tBA \\ &BD = \sqrt{1+t^2}CA - tAC \qquad DB = \sqrt{1+t^2}AC - tCA \\ &B^2 = C^2 \qquad A^2 = D^2 + 2tB^2 \\ &AD = DA \qquad BC = CB. \end{aligned} \tag{2.9}$$

*Case 2a.* $t^2 \neq -1$. The singular polynomials are projectors

$$f_\pm = \pm(\hat{R}_2 - 1 \mp \sqrt{1+t^2})(1+t^2)^{-1/2}/2. \tag{2.10}$$

Their rank is 2 and the corresponding quantum spaces $Q_+(\hat{R}_2)$ and $Q_-(\hat{R}_2)$ are

$$(\sqrt{1+t^2} - t)x_1^2 = x_2^2 \qquad x_1x_2 = x_2x_1 \tag{2.11}$$

and

$$(\sqrt{1+t^2} + t)x_1^2 = -x_2^2 \qquad x_1x_2 = -x_2x_1 \tag{2.12}$$

respectively. The necessary conditions for the invariance of (2.11) and (2.12) under (1.5) are equivalent to (2.9).

*Case 2b.* $t^2 = -1$. Then there is only one singular polynomial $\hat{R}-1$ (up to a regular factor). It is nilpotent, its rank is one and the quantum space is given by the relation

$$tx_1^2 + x_2^2 = 0. \tag{2.13}$$

The conditions for the invariance of (2.13) under (1.5) are weaker than those following from (1.4). Namely, this quantum space is invariant with respect to (1.5) iff

$$CD = -tAB \qquad DC = -tAB \qquad A^2 = D^2 + tB^2 + tC^2. \tag{2.14}$$

*Case 3.*

$$\hat{R}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & st & 0 \\ 0 & s & 0 & 0 \\ 1 & 0 & 0 & -t \end{pmatrix} \qquad s^2 = 1. \tag{2.15}$$

The minimal polynomial condition for $\hat{R}_3$ is

$$(\hat{R}_3 - 1)(\hat{R}_3 + t) = 0. \tag{2.16}$$

The relationships of the algebra $A_R$ are

$$\begin{aligned} &B^2 = 0 \qquad BA = stAB \qquad BD = -stDB \qquad BC = tCB \\ &A^2 = D^2 + (1+t)C^2 \qquad [A,D] = s(1-t)CB \\ &CA = sAC - DB \qquad CD = AB - sDC. \end{aligned} \tag{2.17}$$

*Case 3a.* $t \neq -1$. The singular polynomials are projectors

$$f_+ = (\hat{R}_3 - 1)/(1+t) \qquad f_- = (\hat{R}_3 + t)/(1+t) \tag{2.18}$$

and the quadratic relationships for the quantum spaces $Q_+(\hat{R}_3)$ and $Q_-(\hat{R}_3)$ are

$$x_1^2 = (1+t)x_2^2 \qquad x_1x_2 = sx_2x_1 \tag{2.19}$$

and

$$x_1^2 = 0 \qquad x_1x_2 = -stx_2x_1 \tag{2.20}$$

respectively. These two quantum spaces, or more explicitly, the relationships (2.19) and (2.20) are invariant with respect to (1.5) if and only if the relationships (2.17) of $A_R$ are satisfied.

*Case 3b.* $t = -1$. There is only one singular polynomial $\hat{R}_3 - 1$ and one quantum space given by

$$x_1^2 = 0 \qquad x_1x_2 = sx_2x_1. \tag{2.21}$$

The invariance transformation conditions of this quantum space are

$$B^2 = 0 \qquad BA = -sAB \qquad BD = sDB \qquad [B,C] = s[D,A]. \tag{2.22}$$

*Case 4.* This is equivalent to the preceding one because $R_3 = PR_4P$ [3].

*Case 5.*

$$\hat{R}_5 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-t & qt & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{2.23}$$

This is the most investigated case. For $q = 1/t$ one gets the famous $\mathrm{SL}_q(2)$, for $q \neq t$ (see [5–8]).

The minimal polynomial condition for $\hat{R}_5$ is

$$(\hat{R} - q)(\hat{R} + t) = 0. \tag{2.24}$$

The invariance algebra $A_R$ for $\hat{R}_5$ is given by

$$\begin{aligned} &BA = tAB \qquad AC = qCA \qquad BD = qDB \qquad DC = tCD \\ &BC = qtCB \qquad AD - DA = (q-t)CB. \end{aligned} \tag{2.25}$$

*Case 5a.* $q \neq -t$. There are two singular polynomials

$$f_+ = -(\hat{R} - q)/(q + t) \qquad f_- = (\hat{R} + t)/(q + t) \tag{2.26}$$

with ranks 1 and 3. The quadratic relationships for the quantum spaces $Q_+(\hat{R}_5)$ and $Q_-(\hat{R}_5)$ are

$$x_1 x_2 = q x_2 x_1 \tag{2.27}$$

and

$$x_1^2 = x_2^2 = 0 \qquad x_1 x_2 = -t x_2 x_1 \tag{2.28}$$

respectively. In [1] these quantum spaces are denoted $A_q^{(2,0)}$ and $A_t^{(0,2)}$. Necessary and sufficient conditions for the invariance of $Q_\pm(\hat{R}_5)$ are identical to (2.25).

*Case 5b.* $q = -t$. There is only one singular polynomial, $\hat{R} - q$. It is nilpotent and has rank 1. The quantum space is given by (2.27). The sufficient and necessary conditions for the invariance of this quantum space

$$AC = qCA \qquad BD = qDB \qquad BC - q^2 CB = q[D, A] \tag{2.29}$$

are weaker than the conditions for the algebra $A_R$.

*Case 6.* (See also [9].)

$$\hat{R}_6 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-t & qt & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \qquad q \neq -t. \tag{2.30}$$

The minimal polynomial condition for $\hat{R}_6$ and formulae for the singular polynomials are identical to case 5 but both $f_+$ and $f_-$ have rank 2. The quadratic relationships for the quantum spaces $Q_+(\hat{R}_6)$ and $Q_-(\hat{R}_6)$ are

$$x_1^2 = 0 \qquad x_1 x_2 = q x_2 x_1 \tag{2.31}$$

and

$$x_2^2 = 0 \qquad x_1 x_2 = -t x_2 x_1 \tag{2.32}$$

respectively. The algebra $A_R$ given by

$$\begin{aligned} &BA = tAB \qquad AC = qCA \qquad BD = -tDB \qquad DC = -qCD \\ &BC = qtCB \qquad AD - DA = (q-t)CB \qquad B^2 = C^2 = 0 \end{aligned} \tag{2.33}$$

also represents the necessary and sufficient conditions for the invariance of $Q_\pm(\hat{R}_6)$ under (1.5).

*Case 7.*

$$\hat{R}_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & s & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad s^2 = 1. \tag{2.34}$$

This is one of the most peculiar cases. (We consider solution $R_7$ of [3] with $\varepsilon' = 1$ only because for $\varepsilon' = -1$ $R_7$ is equal to $R_3$ with $t = 1$.)

The matrix $\hat{R}_7$ does not satisfy the Hecke condition but its minimal polynomial is of the third degree:

$$(\hat{R} - 1)^2(\hat{R} + 1) = 0. \tag{2.35}$$

There are four singular polynomials and corresponding quantum spaces:

$$f_1 = \hat{R} + 1 \Rightarrow x_1^2 = 0 \qquad x_2^2 = 0 \qquad x_1 x_2 = -s x_2 x_1 \tag{2.36}$$

$$f_2 = \hat{R} - 1 \Rightarrow x_1^2 = 0 \qquad x_1 x_2 = s x_2 x_1 \tag{2.37}$$

$$f_3 = f_1 f_2 = \hat{R}^2 - 1 \Rightarrow x_1^2 = 0 \tag{2.38}$$

$$f_4 = f_2^2 = (\hat{R} - 1)^2 \Rightarrow x_1 x_2 = s x_2 x_1. \tag{2.39}$$

Only $f_4$ is proportional to a projector. The necessary and sufficient conditions for the invariance of the quantum spaces given by (2.36)–(2.39) are

$$\begin{aligned} &AB = BA = 0 \qquad B^2 = 0 \qquad [A, D] = [B, C] = 0 \\ &AC = sCA \qquad BD = sDB \qquad DC = sCD. \end{aligned} \tag{2.40}$$

The relationships (1.4) of the algebra $A_R$, moreover, require

$$A^2 = D^2 \qquad BD = 0. \tag{2.41}$$

*Case 8.* This corresponds to the diagonal $R$-matrices:

$$\hat{R}_8 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}. \tag{2.42}$$

There are several subcases depending on the values of $q, r, s$ and $t$.

*Case 8a.* $q \neq t$, $q^2 \neq rs$, $t^2 \neq rs$. In this generic case the minimal polynomial for $\hat{R}_8$ is given by the characteristic polynomial

$$(\hat{R} - q)(\hat{R} - t)(\hat{R}^2 - rs) = 0. \tag{2.43}$$

The singular polynomials are

$$f_1 = (\hat{R} - q) \qquad f_2 = (\hat{R} - t) \qquad f_3 = (\hat{R} + \sqrt{rs}) \qquad f_4 = (\hat{R} - \sqrt{rs}) \tag{2.44}$$

and products

$$f_i f_j \qquad f_i f_j f_k \qquad i,j,k = 1,\ldots,4,\ i \neq j \neq k,\ i \neq k.$$

The quantum spaces are then obtained by arbitrary combinations of conditions from the set

$$\{x_1^2 = 0,\ x_2^2 = 0,\ \rho x_1 x_2 + \tau x_2 x_1 = 0,\ \rho x_1 x_2 - \tau x_2 x_1 = 0\} \tag{2.45}$$

where $\rho^2 = r$, $\tau^2 = s$. The determining relationships of the invariance algebra for the generic case are

$$\begin{gathered} AB = BA = AC = CA = BD = DB = CD = DC = 0 \\ AD = DA \qquad rBC = sCB \qquad B^2 = C^2 = 0. \end{gathered} \tag{2.46}$$

They define the invariance transformations for all quantum spaces given previously. The relationships can also be obtained as necessary conditions for $A_R$ to be a common module for quantum spaces

$$Q_1(\hat{R}_{8a}) := \mathbb{C}\langle x_1, x_2\rangle/(x_1^2 = 0) \tag{2.47}$$

$$Q_2(\hat{R}_{8a}) := \mathbb{C}\langle x_1, x_2\rangle/(x_2^2 = 0) \tag{2.48}$$

$$Q_3(\hat{R}_{8a}) := \mathbb{C}\langle x_1, x_2\rangle/(\rho x_1 x_2 + \tau x_2 x_1 = 0) \tag{2.49}$$

$$Q_4(\hat{R}_{8a}) := \mathbb{C}\langle x_1, x_2\rangle/(\rho x_1 x_2 - \tau x_2 x_1 = 0). \tag{2.50}$$

given by projectors proportional to $f_i f_j f_k$.

*Case 8b.* $q = t$, $q^2 \neq rs$. The minimal polynomial condition is

$$(\hat{R} - q)(\hat{R}^2 - rs) = 0. \tag{2.51}$$

The singular polynomials are

$$f_1 = (\hat{R} - q) \qquad f_2 = (\hat{R} + \sqrt{rs}) \qquad f_3 = (\hat{R} - \sqrt{rs}) \tag{2.52}$$

and products $f_1 f_2$, $f_1 f_3$ and $f_2 f_3$. The quantum spaces are obtained by arbitrary combinations of conditions from the set

$$\{x_1^2 = x_2^2 = 0,\ \rho x_1 x_2 + \tau x_2 x_1 = 0,\ \rho x_1 x_2 - \tau x_2 x_1 = 0\}. \tag{2.53}$$

The relationships for $A_R$

$$\begin{gathered} AB = BA = AC = CA = BD = DB = CD = DC = 0 \\ AD = DA \qquad rBC = sCB \end{gathered} \tag{2.54}$$

are equivalent to the condition that the quantum spaces

$$Q_\pm(\hat{R}_{8b}) := \mathbb{C}\langle x_1, x_2\rangle/(\rho x_1 x_2 \pm \tau x_2 x_1 = 0) \tag{2.55}$$

$$Q_0(\hat{R}_{8b}) := \mathbb{C}\langle x_1, x_2\rangle/(x_1^2 = x_2^2 = 0) \tag{2.56}$$

be invariant under (1.5).

*Case 8c.* $q \neq t$, $q^2 \neq t^2 = rs$. The minimal polynomial condition and the singular polynomials are the same as in the case 8b. However, the quantum spaces are obtained by combinations of three other conditions, namely those from the set

$$\{x_1^2 = 0,\ x_2^2 = \rho x_1 x_2 + \tau x_2 x_1 = 0,\ \rho x_1 x_2 - \tau x_2 x_1 = 0\}. \tag{2.57}$$

The relationships for $A_R$

$$\begin{aligned} &AB = BA = AC = CA = 0 \qquad rBD = tDB \qquad tCD = rDC \\ &AD = DA \qquad rBC = sCB \qquad B^2 = C^2 = 0 \end{aligned} \tag{2.58}$$

are equivalent to the condition that the quantum spaces

$$Q_1(\hat{R}_{8c}) := \mathbb{C}\langle x_1, x_2\rangle/(x_1^2 = 0) \tag{2.59}$$

$$Q_2(\hat{R}_{8c}) := \mathbb{C}\langle x_1, x_2\rangle/(x_2^2 = \rho x_1 x_2 + \tau x_2 x_1 = 0) \tag{2.60}$$

$$Q_3(\hat{R}_{8c}) := \mathbb{C}\langle x_1, x_2\rangle/(\rho x_1 x_2 - \tau x_2 x_1 = 0) \tag{2.61}$$

be invariant under (1.5).

We do not consider cases $q = \pm t$, $q^2 = rs$ as they belong to cases 5a and 6.

*Case 9.*

$$\hat{R}_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad t^2 \neq 1. \tag{2.62}$$

(The case $t^2 = 1$ is equivalent upto a similarity transformation to the case 5a.)

The minimal polynomial condition for $\hat{R}_9$ is

$$(\hat{R}^2 - 1)(\hat{R} - t) = 0 \tag{2.63}$$

and the singular polynomials are

$$f_1 = (\hat{R} + 1) \qquad f_2 = (\hat{R} - 1) \qquad f_3 = (\hat{R} - t) \tag{2.64}$$

and products $f_1 f_2$, $f_1 f_3$ and $f_2 f_3$. The quantum spaces are obtained by combinations of conditions from the set

$$\{x_1^2 + x_2^2 = 0,\ x_1^2 - x_2^2 = 0,\ x_1 x_2 = x_2 x_1 = 0\}. \tag{2.65}$$

The determining relationships of the algebra $A_R$

$$\begin{aligned} &AB = BA = AC = CA = BD = DB = CD = DC = 0 \\ &B^2 = C^2 \qquad A^2 = D^2 \end{aligned} \tag{2.66}$$

are necessary and sufficient conditions for invariance of quantum spaces

$$Q_{\pm}(\hat{R}_9) := \mathbb{C}\langle x_1, x_2\rangle/(x_1^2 \pm x_2^2 = 0) \tag{2.67}$$

$$Q_0(\hat{R}_9) := \mathbb{C}\langle x_1, x_2\rangle/(x_1 x_2 = x_2 x_1 = 0) \tag{2.68}$$

under (1.5). The other quantum spaces obtained from (2.65) are invariant with respect to (1.5) as well.

## 3. Conclusions

We have investigated the quantum spaces given by the quadratic relationships of the form (1.2) where the matrices $\hat{R}$ correspond to the list of $4 \times 4$ constant solutions of the YBE [3] and their invariance algebras.

Due to the eight-vertex form of the solutions of the YBE, the defining relationships for the quantum spaces have the form

$$\alpha x_1^2 + \beta x_2^2 = 0 \tag{3.1}$$

or

$$\gamma x_1 x_2 + \delta x_2 x_1 = 0 \tag{3.2}$$

According to (3.1) we can distinguish 'light-cone' quantum spaces, i.e. spaces such that $\alpha\beta \neq 0$ (e.g. $Q_\pm(\hat{R}_1)$, $Q_\pm(\hat{R}_2)$, $Q_+(\hat{R}_3)$ and $Q_\pm(\hat{R}_9)$) and quantum spaces with two, one or none Grassmanian coordinates (e.g. $Q_\pm(\hat{R}_5)$ and $Q_\pm(\hat{R}_6)$).

Most of the solutions of YBE (cases 1–6) presented here satisfy the so-called Hecke condition

$$(\hat{R} - \lambda_+)(\hat{R} - \lambda_-) = 0 \tag{3.3}$$

where $\lambda_\pm$ are eigenvalues of $\hat{R}$. In other words, their minimal polynomials are quadratic. In these cases the only singular polynomials, which define the quantum spaces, are

$$f_\pm = g_\pm(\hat{R} - \lambda_\pm) \tag{3.4}$$

where $g_\pm$ are arbitrary regular polynomials of $\hat{R}$.

For $\lambda_+ \neq \lambda_-$ we can set $g_\pm = \pm 1/(\lambda_- - \lambda_+)$ and due to (3.3)

$$f_\pm^2 = f_\pm \qquad f_+ f_- = 0 \qquad f_+ + f_- = 1 \tag{3.5}$$

so that the $f_\pm$ are projectors that decompose $\mathbb{C}^4$ into two orthogonal subspaces. The conditions for the invariance of $Q_\pm(\hat{R})$ under (1.5) are then equivalent to relationships (1.4). Indeed, the requirement that $Q_\pm(\hat{R})$ are invariant with respect to (1.5) and (3.5) yields

$$f_+(T \otimes T) f_- = 0 \qquad f_-(T \otimes T) f_+ = 0 \tag{3.6}$$

and from (3.4) and (3.6) we get (1.4).

If $\lambda_+ = \lambda_-$ then there is only one singular polynomial (up to a regular factor) so that $Q_+(\hat{R}) = Q_-(\hat{R})$. Conditions for the invariance of this quantum space are then weaker than the relationships defining the algebra $A_R$.

If the minimal polynomial of a matrix $\hat{R}$ is of degree higher than two then there are more than two quantum spaces. The condition that the transformation (1.5) be a homomorphism of all the quantum spaces corresponding to a fixed $\hat{R}$ is not always equivalent to relationships for the matrix algebra $A_R$ (case 7). However, it seems that the equivalence holds in cases when among the singular polynomials $f(\hat{R})$ there are projectors that decompose $\mathbb{C}^4$ into a sum of orthogonal subspaces.

Let us finally note that this approach to the (matrix quadratic) algebras $A_R$ is slightly different from [1] because in [1] one starts with just one quantum space $A$ and then constructs its dual $A^!$ and algebras $end(A)$, $e(A,g)$ (for definitions see [1]). Here we work with all the quantum spaces that by virtue of (1.2) correspond to a given $R$-matrix. The approaches are equivalent if the Hecke condition (3.3) with $\lambda_+ \neq \lambda_-$ holds and

$$\hat{R} = \hat{R}^{\mathrm{T}} \tag{3.7}$$

which implies $Q_+(\hat{R}) = Q_-(\hat{R})^!$.

The relationship (3.7) is fulfilled for $\hat{R}_2$ with any $t$ but in general it imposes restrictions on the $R$-matrices. For $\hat{R}_5$, $\hat{R}_6$ we get $q = 1/t$ (and thus only one-parametric invariance algebras). It is never fulfilled for $\hat{R}_1$ and $\hat{R}_3$. If (3.3), $\lambda_+ \neq \lambda_-$ and (3.7) holds then $A_R = e(Q_+(\hat{R}), g) = e(Q_-(\hat{R}), g)$.

## References


[1] Manin Yu I 1988 Quantum groups and non-commutative geometry (Montreal: CRM, Université Montréal)

[2] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1989 *Algebra i Analiz* **1** 178 (in Russian)

[3] Hlavatý L 1987 *J. Phys. A: Math. Gen.* **20** 1661

[4] Ge M-L, Gwa L-H and Zhao H-K 1990 *J. Phys. A: Math. Gen.* **23** L795

[5] Kulish P P 1990 *Zapiski LOMI* **180** 89 (in Russian)

[6] Sudbery A 1990 *J. Phys. A: Math. Gen.* **23** L697

[7] Schirrmacher A, Wess J and Zumino B 1991 *Z. Phys.* C **49** 317

[8] Burdík Č and Hlavatý L 1991 *J. Phys. A: Math. Gen.* **24** L165

[9] Bednář M, Burdík Č, Hlavatý L and Levinský R 1991 On quantum algebras related to the fermionic algebras *Preprint*